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A CONCENTRIC ARC CRACK IN A CIRCULAR DISK

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Abstract—a circular disk containing a concentric arc crack is studied in this paper using dislocation pile-up and singular integral equation techniques. Related algorithms for computing mode I and mode II stress intensity factors of the curvilinear crack are obtained. Numerical results for several loading cases and various crack geometries are also presented.

1. INTRODUCTION

Disk-crack (disk embedded a crack) problems, as a category of classic LEFM (linear elastic fracture mechanics) cases, have been considered by a number of investigators [e.g. Bowie and Neal (1970); Rooke and Tweed (1972 and 1973a,b); Tweed and Rooke (1972, 1973); Isida (1975); Yaremen (1979); Chang (1983); Xu and Delale (1992); Delale and Xu (1994); Xu (1994) etc.]. Related algorithms have also been developed for computing the stress intensity factors of the crack. In all those studies, however, although the crack could be either an internal one (crack ends do not touch the edge of the disk) or an edge one (one crack end touches the edge of the disk), the crack geometry is limited to a rectilinear one. As some supplementary results, a circular disk containing a concentric arc crack is considered in this paper using dislocation pile-up and singular integral equation techniques. Relevant algorithms for computing mode I and mode II stress intensity factors of the curvilinear crack are presented. Numerical results for several loading cases and various crack geometries are also obtained.

Methodology-wise, Erdogan *et al.* (1974) have addressed the arbitrarily oriented crack cases using the singular integral equation method, whereas this paper furnishes some detailed analysis and treatments of this connection.

2. FORMULATIONS

The general procedures for tackling crack problems with dislocation pile-up and singular integral equation techniques, as exemplified by Erdogan and Gupta (1972), Delale and Erdogan (1982), etc., can be summarized as follows:

(1) First, superpose the original case with the uncracked geometry solution to reduce the problem under consideration to a perturbation case (problem with the same crack configuration but the only loading is the crack-surface self-equilibrating pressure and shear stress). As far as the singular nature of the crack is concerned, the perturbation case is the same as the original one. Next, using the corresponding dislocation solution as a basic solution or Green's function and by integration, derive a set of Cauchy-type singular integral equations.

(2) The singularity and the dominant terms of the near-crack-tip stresses can be determined through an asymptotic analysis of the integrals in the preceding integral equations; therefore, the stress intensity factors of the crack may be formulated in terms of the solution to the singular integral equation and calculated numerically with a collocation method to a satisfactory accuracy.

In what follows, a circular disk of radius R containing a concentric arc crack ($r = \rho$, $\theta_1 < \theta < \theta_2$) is considered. The dislocation solution for the disk-crack problems has been obtained by the present author (Xu, 1994) and the major results can be found in the Appendix.

Following the procedures, the original case is reduced to a perturbation one, the boundary conditions of which can be written as below:

$$\sigma_{\theta\theta}(R,\theta) = \tau_{r\theta}(R,\theta) = 0, \quad 0 \leqslant \theta \leqslant 2\pi \tag{1}$$

$$\sigma_{rr}(\rho,\theta) = p(\theta), \quad (\theta_1 < \theta < \theta_2) \tag{2}$$

$$\tau_{r\theta}(\rho,\theta) = q(\theta), \quad (\theta_1 < \theta < \theta_2), \tag{3}$$

where $p(\theta) = -\sigma^*_{rr}(\rho,\theta)$ and $q(\theta) = -\tau^*_{r\theta}(\rho,\theta)$, σ^*_{rr} and $\tau^*_{r\theta}$ are radial and shear stresses of the uncracked geometry solution (the problem with the same load and geometry except without a crack). If we define

$$f(x) = \frac{\partial[u_y(x,y^+)-u_y(x,y^-)]}{\partial x}, \quad (x_{b1} < x < x_{a1})$$

$$g(x) = \frac{\partial[u_x(x,y^+)-u_x(x,y^-)]}{\partial x}, \quad (x_{b1} < x < x_{a1}), \tag{4}$$

where u_x and u_y are the cartesian coordinate displacements, and x and y are cartesian coordinates of the point on the crack; y^+ and y^- denote the point on the upper and lower edges of the crack, respectively, and x_{a1} and x_{b1} are the projections of a_1 and b_1 (crack ends) on the x axis (see Fig. 1).

By integrating the dislocation solutions and imposing the boundary conditions (2) and (3), we may arrive at

$$-\int_{\theta_1}^{\theta_2} f_1(\alpha)c\tan\frac{\theta-\alpha}{2}\sin\theta\sin\alpha\,\mathrm{d}\alpha+\int_{\theta_1}^{\theta_2}\rho f_1(\alpha)k_1(\theta,\alpha)\sin\alpha\,\mathrm{d}\alpha$$

$$-\int_{\theta_1}^{\theta_2} g_1(\alpha)c\tan\frac{\theta-\alpha}{2}\cos\theta\sin\alpha\,\mathrm{d}\alpha+\int_{\theta_1}^{\theta_2}\rho g_1(\alpha)k_2(\theta,\alpha)\sin\alpha\,\mathrm{d}\alpha = \frac{\pi(1+\kappa)}{\mu}p(\theta),$$

$$-\int_{\theta_1}^{\theta_2} f_1(\alpha)c\tan\frac{\theta-\alpha}{2}\frac{(\cos\theta+\cos\alpha)}{2}\sin\alpha\,\mathrm{d}\alpha+\int_{\theta_1}^{\theta_2}\rho f_1(\alpha)k_3(\theta,\alpha)\sin\alpha\,\mathrm{d}\alpha$$

$$+\int_{\theta_1}^{\theta_2} g_1(\alpha)c\tan\frac{\theta-\alpha}{2}\frac{(\sin\theta+\sin\alpha)}{2}\sin\alpha\,\mathrm{d}\alpha+\int_{\theta_1}^{\theta_2}\rho g_1(\alpha)k_4(\theta,\alpha)\sin\alpha\,\mathrm{d}\alpha = \frac{\pi(1+\kappa)}{\mu}q(\theta),$$

$$(\theta_1 < \theta < \theta_2), \tag{5}$$

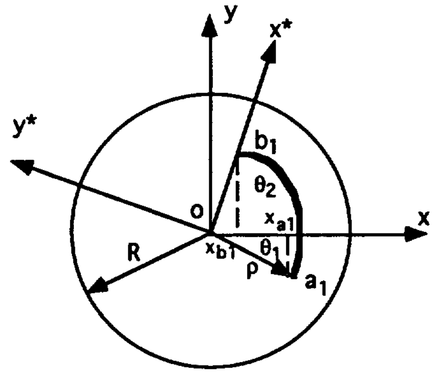

Fig. 1. Circular disk containing a concentric arc crack.

where

$$f_1(\alpha) = f(\rho\cos\alpha), \quad g_1(\alpha) = g(\rho\cos\alpha) \tag{6}$$

and

$$\begin{aligned} k_1(\theta,\alpha) = \Bigg\{ & -\frac{1}{\rho}\cos\alpha - \frac{2\rho\cos\alpha}{R^2} + \frac{2\rho R^2\cos\alpha - \rho^3\cos\theta - \rho R^2\cos(\alpha-2\theta)}{R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)} \\ & + \frac{\rho R^4(3\rho^2-R^2)\cos(\alpha-2\theta) - R^2\rho^3(5R^2+6\rho^2)\cos(2\alpha-\theta) + \rho^3R^2(2\rho^2+R^2)\cos(3\alpha-2\theta)}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & + \frac{\rho[2R^6+4\rho^6+8\rho^2R^4+R^2\rho^4)\cos\alpha - \rho(4R^6+6\rho^4R^2+\rho^6-2\rho^2R^4)\cos\theta}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & - \frac{2R^2(R^2-\rho^2)[\rho R^6\cos(\alpha-2\theta)+\rho^7\cos(3\alpha-2\theta)-\rho^5(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \\ & - \frac{2R^2(R^2-\rho^2)[3R^2\rho^3(R^2+\rho^2)\cos\alpha - R^4\rho(R^2+3\rho^2)\cos\theta]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \Bigg\} \end{aligned} \tag{7}$$

$$\begin{aligned} k_2(\theta,\alpha) = \Bigg\{ & \frac{1}{\rho}\sin\alpha + \frac{2\rho\sin\alpha}{R^2} + \frac{-2\rho R^2\sin\alpha + \rho^3\sin\theta - \rho R^2\sin(\alpha-2\theta)}{R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)} \\ & + \frac{\rho R^4(3\rho^2-R^2)\sin(\alpha-2\theta) + R^2\rho^3(5R^2+6\rho^2)\sin(2\alpha-\theta) - \rho^3R^2(2\rho^2+R^2)\sin(3\alpha-2\theta)}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & + \frac{\rho(4R^6+6\rho^4R^2+\rho^6-2\rho^2R^4)\sin\theta - \rho(2R^6+8\rho^2R^4+\rho^4R^2+4\rho^6)\sin\alpha}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & - \frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta)-\rho^7\sin(3\alpha-2\theta)+\rho^5(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \\ & - \frac{2R^2(R^2-\rho^2)[R^4\rho(R^2+3\rho^2)\sin\theta - 3R^2\rho^3(R^2+\rho^2)\sin\alpha]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \Bigg\} \end{aligned} \tag{8}$$

$$\begin{aligned} k_3(\theta,\alpha) = \Bigg\{ & \frac{\sin\theta-\sin\alpha}{2\rho} - \frac{\rho^3\sin\theta + \rho R^2\sin(\alpha-2\theta)}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]} \\ & - \frac{\rho R^4(R^2-\rho^2)\sin(\alpha-2\theta) + R^2\rho^3(R^2+2\rho^2)\sin(2\alpha-\theta) - \rho^3R^4\sin(3\alpha-2\theta)}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & - \frac{\rho^3(2R^4+\rho^4-2\rho^2R^2)\sin\theta - 3\rho^5R^2\sin\alpha}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2} \\ & - \frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta)-\rho^7\sin(3\alpha-2\theta)+\rho^5(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \\ & - \frac{2R^2(R^2-\rho^2)[R^4\rho(R^2+3\rho^2)\sin\theta - 3R^2\rho^3(R^2+\rho^2)\sin\alpha]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3} \Bigg\} \end{aligned} \tag{9}$$

$$k_4(\theta,\alpha) = \left\{\frac{\cos\theta-\cos\alpha}{2\rho} + \frac{-\rho^3\cos\theta+\rho R^2\cos(\alpha-2\theta)}{R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)}\right.$$

$$-\frac{\rho R^4(\rho^2-R^2)\cos(\alpha-2\theta)+R^2\rho^3(R^2+2\rho^2)\cos(2\alpha-\theta)-\rho^3R^4\cos(3\alpha-2\theta)}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2}$$

$$-\frac{\rho^3(2R^4+\rho^4-2\rho^2R^2)\cos\theta-3\rho^5R^2\cos\alpha}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^2}$$

$$-\frac{2R^2(R^2-\rho^2)[-\rho R^6\cos(\alpha-2\theta)-\rho^7\cos(3\alpha-2\theta)+\rho^5(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3}$$

$$\left.-\frac{2R^2(R^2-\rho^2)[R^4\rho(R^2+3\rho^2)\cos\theta-3R^2\rho^3(R^2+\rho^2)\cos\alpha]}{[R^4+\rho^4-2\rho^2R^2\cos(\alpha-\theta)]^3}\right\}. \quad (10)$$

Introducing the transformation

$$s = \tan\frac{\theta}{2}, \quad t = \tan\frac{\alpha}{2}, \quad (11)$$

hence

$$\mathrm{d}\alpha = \frac{2\,\mathrm{d}t}{1+t^2}, \quad c\tan\frac{\theta-\alpha}{2} = \frac{1+st}{s-t} \quad (12)$$

Let

$$c_1 = \tan\frac{\theta_1}{2}, \quad d_1 = \tan\frac{\theta_2}{2}$$

$$P(s) = p(2\tan^{-1}s), \quad Q(s) = q(2\tan^{-1}s)$$

$$f_2(t) = f_1(2\tan^{-1}t), \quad g_2(t) = g_1(2\tan^{-1}t).$$

With some algebra, we may obtain

$$-\int_{c_1}^{d_1}\frac{f_2(t)s}{t-s}\mathrm{d}t + \int_{c_1}^{d_1} f_2(t)K_1(s,t)\,\mathrm{d}t + \int_{c_1}^{d_1}\frac{g_2(t)(1-s^2)}{t-s}\mathrm{d}t - \int_{c_1}^{d_1} g_2(t)K_2(s,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{\mu}P(s) \quad (13)$$

$$-\int_{c_1}^{d_1}\frac{f_2(t)}{t-s}\frac{1+s^2}{4}\left(\frac{1-t^2}{1+t^2}+\frac{1-s^2}{1+s^2}\right)\mathrm{d}t + \int_{c_1}^{d_1} f_2(t)K_3(s,t)\,\mathrm{d}t - \int_{c_1}^{d_1}\frac{g_2(t)(1+s^2)}{t-s}\left(\frac{t}{1+t^2}+\frac{s}{1+s^2}\right)\mathrm{d}t - \int_{c_1}^{d_1} g_2(t)K_4(s,t)\,\mathrm{d}t = \frac{\pi(1+\kappa)}{\mu}Q(s),$$

$$(c_1 < s < d_1), \quad (14)$$

where

$$K_1(s,t) = -\left[\frac{s^2}{1+s^2} + \frac{\rho}{2}k_1(2\tan^{-1}s, 2\tan^{-1}t)\right] \tag{15}$$

$$K_2(s,t) = -\left[\frac{s(1-s^2)}{1+s^2} + \rho k_2(2\tan^{-1}s, 2\tan^{-1}t)\right] \tag{16}$$

$$K_3(s,t) = -\left[\frac{s}{4}\left(\frac{1-t^2}{1+t^2} + \frac{1-s^2}{1+s^2}\right) + \frac{\rho}{2}k_3(2\tan^{-1}s, 2\tan^{-1}t)\right] \tag{17}$$

$$K_4(s,t) = \left[s\left(\frac{t}{1+t^2} + \frac{s}{1+s^2}\right) - \rho k_4(2\tan^{-1}s, 2\tan^{-1}t)\right]. \tag{18}$$

For an embedded crack, due to the single-valuedness of the displacements at the crack ends, it follows that

$$\int_{c_1}^{d_1} f_2(t)\,\mathrm{d}t = 0 \tag{19}$$

and

$$\int_{c_1}^{d_1} g_2(t)\,\mathrm{d}t = 0. \tag{20}$$

Examining the kernel functions $K_i(s,t)$, $i = 1$–4, it may be readily found that they are all bounded at $[c_1, d_1]$, i.e. they are all Fredholm kernels. Therefore, the only singularities come from those Cauchy kernel terms. To evaluate the singularities, we assume

$$f_2(t) = \frac{F_2(t)}{(t-c_1)^{\alpha_1}(d_1-t)^{\beta_1}} = \frac{F_2(t)}{(t-c_1)^{\alpha_1}(t-d_1)^{\beta_1}\,\mathrm{e}^{-\pi i\beta_1}}, \quad (c_1 < t < d_1) \tag{21}$$

$$g_2(t) = \frac{G_2(t)}{(t-c_1)^{\alpha_2}(d_1-t)^{\beta_2}} = \frac{G_2(t)}{(t-c_1)^{\alpha_2}(t-d_1)^{\beta_2}\,\mathrm{e}^{-\pi i\beta_2}}, \quad (c_1 < t < d_1) \tag{22}$$

where

$$0 < \mathrm{Re}\{\alpha_1\} < 1, \quad 0 < \mathrm{Re}\{\alpha_2\} < 1, \quad 0 < \mathrm{Re}\{\beta_1\} < 1, \quad 0 < \mathrm{Re}\{\beta_2\} < 1.$$

$F_2(t)$ and $G_2(t)$ are continuous functions satisfying $F_2(c_1) \neq 0$, $F_2(d_1) \neq 0$, $G_2(c_1) \neq 0$, $G_2(d_1) \neq 0$.

Using the Plemeji formula (Muskhelishvili, 1953a), we may expand the integrals in eqns (13) and (14) as follows:

$$\int_{c_1}^{d_1} \frac{f_2(t)s}{t-s}\,\mathrm{d}t = \frac{\pi F_2(c_1)c_1}{(d_1-c_1)^{\beta_1}}\frac{c\tan(\alpha_1\pi)}{(s-c_1)^{\alpha_1}} - \frac{\pi F_2(d_1)d_1}{(d_1-c_1)^{\alpha_1}}\frac{c\tan(\beta_1\pi)}{(d_1-s)^{\beta_1}} + F_0(s) \tag{23}$$

$$\int_{c_1}^{d_1} \frac{f_2(t)}{t-s}\frac{(1+s^2)}{4}\left(\frac{(1-t^2)}{1+t^2} + \frac{(1-s^2)}{1+s^2}\right)\mathrm{d}t = \frac{\pi F_2(c_1)}{(d_1-c_1)^{\beta_1}}\frac{(1-c_1^2)}{2}\frac{c\tan(\alpha_1\pi)}{(s-c_1)^{\alpha_1}}$$
$$- \frac{\pi F_2(d_1)}{(d_1-c_1)^{\alpha_1}}\frac{(1-d_1^2)}{2}\frac{c\tan(\beta_1\pi)}{(d_1-s)^{\beta_1}} + F_1(s) \tag{24}$$

$$\int_{c_1}^{d_1} \frac{g_2(t)(1-s^2)}{t-s}\,\mathrm{d}t = \frac{\pi G_2(c_1)(1-c_1^2)c\tan(\alpha_2\pi)}{(d_1-c_1)^{\beta_2}}\frac{1}{(s-c_1)^{\alpha_2}} - \frac{\pi G_2(d_1)(1-d_1^2)c\tan(\beta_2\pi)}{(d_1-c_1)^{\alpha_2}}\frac{1}{(d_1-s)^{\beta_2}} + G_0(s) \quad (25)$$

$$\int_{c_1}^{d_1} \frac{g_2(t)(1-s^2)}{t-s}\left(\frac{t}{1+t^2}+\frac{s}{1+s^2}\right)\mathrm{d}t = \frac{\pi G_2(c_1)c_1}{(d_1-c_1)^{\beta_2}}\frac{2(1-c_1^2)}{(1+c_1^2)}\frac{c\tan(\alpha_2\pi)}{(s-c_1)^{\alpha_2}} - \frac{\pi G_2(d_1)d_1}{(d_1-c_1)^{\alpha_2}}\frac{2(1-d_1^2)}{(1+d_1^2)}\frac{c\tan(\beta_2\pi)}{(d_1-s)^{\beta_2}} + G_1(s), \quad (26)$$

where F_0, F_1, G_0 and G_1 are sectionally holomorphic functions bounded everywhere, except possibly at the ends, near which

$$|F_k(s)| < \frac{C_k}{(s-c_1)^{p_k}},\quad p_k < \mathrm{Re}\{\alpha_1\},\quad |F_k(s)| < \frac{D_k}{(d_1-s)^{q_k}},\quad q_k < \mathrm{Re}\{\beta_1\},\quad (k=0,1) \quad (27)$$

$$|G_k(s)| < \frac{H_k}{(s-c_1)^{h_k}},\quad h_k < \mathrm{Re}\{\alpha_2\},\quad |G_k(s)| < \frac{S_k}{(d_1-s)^{s_k}},\quad s_k < \mathrm{Re}\{\beta_2\},\quad (k=0,1), \quad (28)$$

where C_k, D_k, H_k and S_k $(k = 0, 1)$ are all real numbers.

Substituting eqns (23)–(26) into eqns (13) and (14), rearranging and mutiplying both sides of the equations by $(s-c_1)^{\alpha_1}$ or $(d_1-s)^{\beta_1}$ (or $(s-c_1)^{\alpha_2}$ or $(d_1-s)^{\beta_2}$) then taking the limit at c_1 or d_1, we obtain

$$-\frac{\pi F_2(c_1)(1+4c_1-c_1^4)}{4(d_1-c_1)^{\beta_1}}c\tan(\alpha_1\pi) = 0 \quad (29)$$

$$\frac{\pi F_2(d_1)(1+4d_1-d_1^4)}{4(d_1-c_1)^{\alpha_1}}c\tan(\beta_1\pi) = 0 \quad (30)$$

$$\frac{\pi G_2(c_1)(1+4c_1-c_1^4)}{4(d_1-c_1)^{\beta_2}}c\tan(\alpha_2\pi) = 0 \quad (31)$$

$$-\frac{\pi G_2(d_1)(1+4d_1-d_1^4)}{4(d_1-c_1)^{\alpha_2}}c\tan(\beta_2\pi) = 0. \quad (32)$$

The solutions of eqns (29)–(32) yield $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5$, which is a known result.

3. STRESS INTENSITY FACTORS

Introducing a coordinate system x^*oy^* (associated polar coordinates are r^*, θ^*), which can be obtained by rotating the original coordinate system $xoy(r, \theta)$ about the z axis through an angle, e.g. for a consideration of crack end, b_1, we may rotate xoy system through θ_2, as shown in Fig. 1. Obviously,

$$r^* = r, \quad \theta^* = \theta - \theta_2 \tag{33}$$

$$\sigma^*_{rr}(r^*, \theta^*) = \sigma_{rr}(r, \theta), \tag{34}$$

where $\sigma^*_{rr}(r^*, \theta^*)$ and $\sigma_{rr}(r, \theta)$ are the radial stress components in the x^*Oy^* and xOy systems, respectively. The mode I stress intensity factor at the crack end, b_1, can be calculated according to the conventional definition as

$$k_1(b_1) = \lim_{\theta^* \to 0^+} \sqrt{2\rho \sin \theta^*}\sigma^*_{rr}(\rho, \theta^*) \tag{35}$$

or

$$k_1(b_1) = \lim_{\theta \to \theta_2^+} \sqrt{2\rho \sin(\theta - \theta_2)}\sigma_{rr}(\rho, \theta) \tag{36}$$

or

$$\begin{aligned} k_1(b_1) &= \lim_{s \to d_1^+} \sqrt{\frac{4\rho(s-d_1)}{1+d_1^2}}\sigma_{rr}(\rho, 2\tan^{-1} s) \\ &= \frac{2\mu}{1+\kappa} \lim_{s \to d_1^-} \sqrt{\frac{\rho}{1+d_1^2}}\sqrt{d_1 - s}[d_1 f_2(s) - (1-d_1^2) g_2(s)] \\ &= \frac{2\mu}{1+\kappa}\sqrt{\frac{\rho}{1+d_1^2}}\left[\frac{d_1 F_2(d_1)}{\sqrt{d_1 - c_1}} - \frac{(1-d_1^2)G_2(d_1)}{\sqrt{d_1 - c_1}}\right]. \end{aligned} \tag{37}$$

In a similar analysis, we arrive at

$$k_1(a_1) = \frac{2\mu}{1+\kappa}\sqrt{\frac{\rho}{1+c_1^2}}\left[-\frac{c_1 F_2(c_1)}{\sqrt{d_1 - c_1}} + \frac{(1-c_1^2)G_2(c_1)}{\sqrt{d_1 - c_1}}\right] \tag{38}$$

$$k_2(a_1) = -\frac{2\mu}{1+\kappa}\sqrt{\frac{\rho}{1+c_1^2}}\left[\frac{(1-c_1^2)F_2(c_1)}{2\sqrt{d_1 - c_1}} + \frac{2c_1 G_2(c_1)}{\sqrt{d_1 - c_1}}\right] \tag{39}$$

$$k_2(b_1) = \frac{2\mu}{1+\kappa}\sqrt{\frac{\rho}{1+d_1^2}}\left[\frac{(1-d_1^2)F_2(d_1)}{2\sqrt{d_1 - c_1}} + \frac{2d_1 G_2(d_1)}{\sqrt{d_1 - c_1}}\right]. \tag{40}$$

4. NUMERICAL PROCEDURES

The singular integral equations (13), (14), (19) and (20) can be solved by a collocation method. Introducing transformations as below,

$$\begin{aligned} s &= \frac{d_1 - c_1}{2}\gamma + \frac{d_1 + c_1}{2}, \quad \text{when } c_1 < s < d_1, \quad -1 < \gamma < 1 \\ t &= \frac{d_1 - c_1}{2}\tau + \frac{d_1 + c_1}{2}, \quad \text{when } c_1 < t < d_1, \quad -1 < \tau < 1 \end{aligned} \tag{41}$$

and letting

$$f_2^*(\tau)=f_2(t),\quad g_2^*(\tau)=g_2(t),\quad P^*(\gamma)=P(s),\quad Q^*(\gamma)=Q(s),$$

$$K_1^*(\gamma,\tau)=K_1(s,t),\quad K_2^*(\gamma,\tau)=K_2(s,t),\quad K_3^*(\gamma,\tau)=K_3(s,t),\quad K_4^*(\gamma,\tau)=K_4(s,t),\tag{42}$$

eqns (13), (14), (19) and (20) become

$$-\int_{-1}^{1}\frac{f_2^*(\tau)}{\tau-\gamma}\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1+c_1}{2}\right)\mathrm{d}\tau+\frac{d_1-c_1}{2}\times\int_{-1}^{1}f_2^*(\tau)K_1^*(\gamma,\tau)\,\mathrm{d}\tau+\int_{-1}^{1}\frac{g_2^*(\tau)\left[1-\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1+c_1}{2}\right)^2\right]}{\tau-\gamma}\mathrm{d}\tau-\frac{d_1-c_1}{2}\int_{-1}^{1}g_2^*(\tau)K_2^*(\gamma,\tau)\,\mathrm{d}\tau=\frac{\pi(1+\kappa)}{\mu}P^*(\gamma)\tag{43}$$

$$-\int_{-1}^{1}\frac{f_2^*(\tau)}{\tau-\gamma}\frac{1+\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}\right)^2}{4}\left[\frac{1-\left(\frac{d_1-c_1}{2}\tau+\frac{d_1-c_1}{2}\right)^2}{1+\left(\frac{d_1-c_1}{2}\tau+\frac{d_1-c_1}{2}\right)^2}+\frac{1-\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}\right)^2}{1+\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}\right)^2}\right]\mathrm{d}t+\frac{d_1-c_1}{2}\int_{-1}^{1}f_2^*(t)K_3^*(\gamma,\tau)\,\mathrm{d}\tau$$
$$-\int_{-1}^{1}\frac{g_2^*(\tau)\left[1+\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}\right)^2\right]}{\tau-\gamma}\left[\frac{\frac{d_1-c_1}{2}\tau+\frac{d_1-c_1}{2}}{1+\left(\frac{d_1-c_1}{2}\tau+\frac{d_1-c_1}{2}\right)^2}+\frac{\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}}{1+\left(\frac{d_1-c_1}{2}\gamma+\frac{d_1-c_1}{2}\right)^2}\right]\mathrm{d}\tau-\frac{d_1-c_1}{2}\int_{-1}^{1}g_2^*(\tau)K_4^*(\gamma,\tau)\,\mathrm{d}\tau=\frac{\pi(1+\kappa)}{\mu}Q^*(\gamma)$$
$$(-1<\gamma<1)\tag{44}$$

$$\int_{-1}^{1}f_2^*(\tau)\,\mathrm{d}\tau=0\tag{45}$$

$$\int_{-1}^{1}g_2^*(\tau)\,\mathrm{d}\tau=0.\tag{46}$$

Accordingly, we have

$$f_2^*(\tau)=\frac{F_2^*(\tau)}{\sqrt{(1-\tau^2)}},\quad(-1<\tau<1)\tag{47}$$

and

$$g_2^*(\tau) = \frac{G_2^*(\tau)}{\sqrt{(1-\tau^2)}}, \quad (-1 < \tau < 1). \tag{48}$$

The stress intensity factors of the crack can be calculated by

$$k_1(a_1) = \frac{\mu}{1+\kappa}\sqrt{\frac{\rho}{1+c_1^2}}\sqrt{d_1-c_1}\,[-c_1F_2^*(-1)+(1-c_1^2)G_2^*(-1)] \tag{49}$$

$$k_1(b_1) = \frac{\mu}{1+\kappa}\sqrt{\frac{\rho}{1+d_1^2}}\sqrt{d_1-c_1}\,[d_1F_2^*(1)-(1-d_1^2)G_2^*(1)] \tag{50}$$

$$k_2(a_1) = -\frac{\mu}{1+\kappa}\sqrt{\frac{\rho}{1+c_1^2}}\sqrt{d_1-c_1}\left[\frac{(1-c_1^2)F_2^*(-1)+4c_1G_2^*(-1)}{2}\right] \tag{51}$$

$$k_2(b_1) = \frac{\mu}{1+\kappa}\sqrt{\frac{\rho}{1+d_1^2}}\sqrt{d_1-c_1}\left[\frac{(1-d_1^2)F_2^*(1)+4d_1G_2^*(1)}{2}\right]. \tag{52}$$

Referring to Erdogan and Gupta (1972) and Iaokimidas and Theocaris (1980), eqns (43)–(46) can be discretized into a set of $2N$ simultaneous algebraic equations,

$$\sum_{i=1}^{n}\left[-\lambda_i\frac{F_2^*(\tau_i)}{\tau_i-\gamma_j}\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)+\lambda_i\frac{d_1-c_1}{2}K_1^*(\gamma_j,\tau_i)F_2^*(\tau_i)\right.$$
$$\left.+\lambda_i\frac{G_2^*(\tau_i)}{\tau_i-\gamma_j}\left[1-\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2\right]-\lambda_i\frac{d_1-c_1}{2}K_2^*(\gamma_j,\tau_i)G_2^*(\tau_i)\right]=\frac{\pi(1+\kappa)}{\mu}P^*(\gamma_j)$$

$$\sum_{i=1}^{n}\left[-\lambda_i\frac{F_2^*(\tau_i)}{\tau_i-\gamma_j}\frac{1+\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2}{4}\left[\frac{1-\left(\frac{d_1-c_1}{2}\tau_i+\frac{d_1+c_1}{2}\right)^2}{1+\left(\frac{d_1-c_1}{2}\tau_i+\frac{d_1+c_1}{2}\right)^2}\right.\right.$$
$$\left.+\frac{1-\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2}{1+\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2}\right]+\lambda_i\frac{d_1-c_1}{2}K_3^*(\gamma_j,\tau_i)F_2^*(\tau_i)-\lambda_i$$
$$\times\frac{G_2^*(\tau_i)\left[1-\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2\right]}{\tau_i-\gamma_j}\left[\frac{\frac{d_1-c_1}{2}\tau_i+\frac{d_1+c_1}{2}}{1+\left(\frac{d_1-c_1}{2}\tau_i+\frac{d_1+c_1}{2}\right)^2}\right.$$
$$\left.+\frac{\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}}{1+\left(\frac{d_1-c_1}{2}\gamma_j+\frac{d_1+c_1}{2}\right)^2}\right]$$

$$-\lambda_i \frac{d_1-c_1}{2} K_4^*(\gamma_j,\tau_i)G_2^*(\tau_i) \Bigg] = \frac{\pi(1+\kappa)}{\mu} Q^*(\gamma_j), \quad (j = 1,2,\ldots,n-1)$$

$$\sum_{i=1}^{n} \lambda_i F_2^*(\tau_i) = 0$$

$$\sum_{i=1}^{n} \lambda_i G_2^*(\tau_i) = 0 \tag{53}$$

where the weight coefficients λ_i are given by

$$\lambda_i = \begin{cases} \dfrac{\pi}{2(n-1)}, & \text{when } i = 1 \text{ or } n \\ \dfrac{\pi}{(n-1)}, & \text{when } i = 2,3,\ldots,n-1 \end{cases} \tag{54}$$

and τ_i and s_j satisfy

$$\begin{aligned} &T_{n-1}(s_j) = 0, \quad j = 1,2,3,\ldots,n-1 \\ &U_{n-2}(\tau_i) = 0, \quad i = 1,2,3,\ldots,n-2 \text{ and } \tau = \pm 1, \end{aligned} \tag{55}$$

where $T_{n-1}(x)$ and $U_{n-2}(x)$ are the Chebyshev polynomials of first and second kinds, respectively.

5. RESULTS AND DISCUSSIONS

Numerical procedures have been implemented to computerize the stress intensity factors of the arc crack for several load cases, as follows.

(a) Uniform distributed load, q_0, acts on the circumference of the disk, i.e. in eqns (13) and (14),

$$\begin{aligned} p(\theta) &= -q_0 \\ q(\theta) &= 0 \end{aligned} \tag{56}$$

(b) Disk rotates with constant angular velocity, ω,

$$\begin{aligned} p(\theta) &= -\frac{1+v}{8} \mathrm{d}\omega^2 R^2 \left(1 - \frac{\rho^2}{R^2}\right) \\ q(\theta) &= 0 \end{aligned} \tag{57}$$

where d is the density, and v is the Poisson's ratio.

(c) Point forces, P, are applid at $\theta = 0$ and $\theta = \pi$ (see Fig. 7),

$$p(\theta) = -\frac{2P}{\pi}\left\{\cos^2\theta\left[\frac{(R-\rho\cos\theta)^3}{[(R-\rho\cos\theta)^2+\rho^2\sin^2\theta]^2} + \frac{(R+\rho\cos\theta)^3}{[(R+\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right]\right.$$

$$+\sin^2\theta\left[\frac{\rho^2\sin^2\theta(R-\rho\cos\theta)}{[(R-\rho\cos\theta)^2+\rho^2\sin^2\theta]^2} + \frac{\rho^2\sin^2(R+\rho\cos\theta)}{[(R+\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right]$$

$$-\sin 2\theta\left[\frac{\rho\sin\theta(R-\rho\cos\theta)^2}{[(R-\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}-\frac{\rho\sin\theta(R+\rho\cos\theta)^2}{[(R+\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right]\Bigg\}+\frac{P}{\pi R}$$

$$q(\theta) = -\frac{2P}{\pi}\Bigg\{\frac{\sin 2\theta}{2}\left[\frac{\rho^2\sin^2\theta(R-\rho\cos\theta)-(R-\rho\cos\theta)^3}{[(R-\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right.$$

$$\left.+\frac{\rho^2\sin^2\theta(R+\rho\cos\theta)-(R+\rho\cos\theta)^3}{[(R+\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right]$$

$$-\cos 2\theta\left[\frac{\rho\sin\theta(R-\rho\cos\theta)^2}{[(R-\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}-\frac{\rho\sin\theta(R+\rho\cos\theta)^2}{[(R+\rho\cos\theta)^2+\rho^2\sin^2\theta]^2}\right]\Bigg\}. \quad (58)$$

Normalized stress intensity factors *versus* the half arc angle $\alpha(\theta_1 = -\alpha, \theta_2 = \alpha)$, for different R/ρ ratios for all the loading cases are obtained and presented in Figs 2–7. Among the results, Figs 2 and 3 are for the distributed load case, and the normalized stress intensity factors are defined as

$$\frac{k_1(a_1)}{q_0\sqrt{\rho}}\left(\frac{k_1(b_1)}{q_0\sqrt{\rho}}\right) \text{ and } \frac{k_2(a_1)}{q_0\sqrt{\rho}}\left(\frac{k_2(b_1)}{q_0\sqrt{\rho}}\right);$$

Figs 4 and 7 are for the rotating disk case, and the normalized stress intensity factors are defined as

$$\frac{k_1(a_1)}{\bar{\sigma}\sqrt{\rho}}\left(\frac{k_1(b_1)}{\bar{\sigma}\sqrt{\rho}}\right) \text{ and } \frac{k_2(a_1)}{\bar{\sigma}\sqrt{\rho}}\left(\frac{k_2(b_1)}{\bar{\sigma}\sqrt{\rho}}\right),$$

where $\bar{\sigma} = \mathrm{d}\omega^2R^2$; Figs 6 and 7 are for the point forces case, and the stress intensity factors are normalized as

$$\frac{k_1(a_1)}{(P/R)\sqrt{\rho}}\left(\frac{k_1(b_1)}{(P/R)\sqrt{\rho}}\right) \text{ and } \frac{k_2(a_1)}{(P/R)\sqrt{\rho}}\left(\frac{k_2(b_1)}{(P/R)\sqrt{\rho}}\right).$$

It may be seen that all those graphs have a similar shape, i.e. they monotonically increase

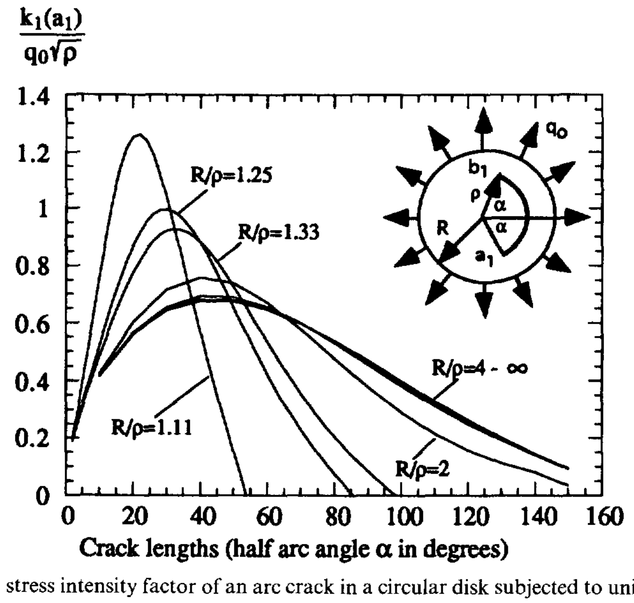

Fig. 2. Mode I stress intensity factor of an arc crack in a circular disk subjected to uniform traction.

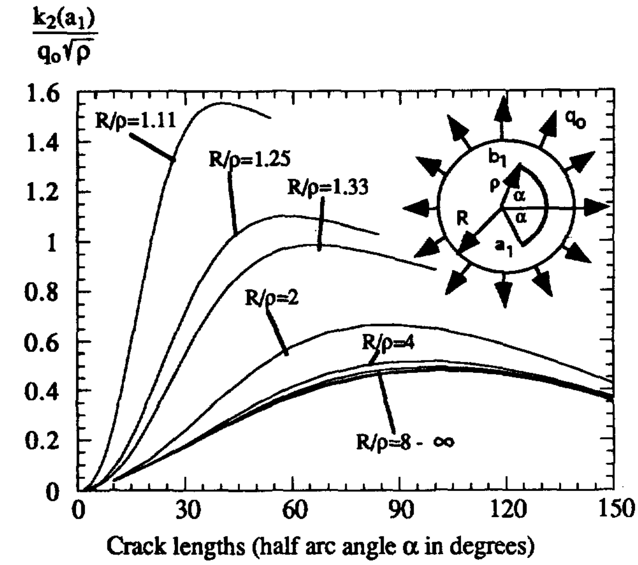

Fig. 3. Mode II stress intensity factor of an arc crack in a circular disk subjected to uniform traction.

until they reach a peak value, then monotonically decrease to a small value. This is quite different from the radial crack case (circular disk having a radial crack), for which [e.g. for distributed load and rotating disk cases, see Rooke and Tweed (1972), Bowie and Neal (1970)], the stress intensity factors increase monotonically with the crack length, which means that once a radial crack propagation coalesces, it will not stop until the disk is completely ruptured.

As a special case, if $R \to \infty$, the considered case degenerates to an infinite plane containing an arc crack and the kernel functions in the integral equations (15)–(18) reduce to

$$K_1(s,t) = -\left[\frac{s^2}{1+s^2} - \frac{1-t^2}{2(1+t^2)}\right] \tag{59}$$

$$K_2(s,t) = -\left[\frac{s(1-s^2)}{1+s^2} + \frac{2t}{1+t^2}\right] \tag{60}$$

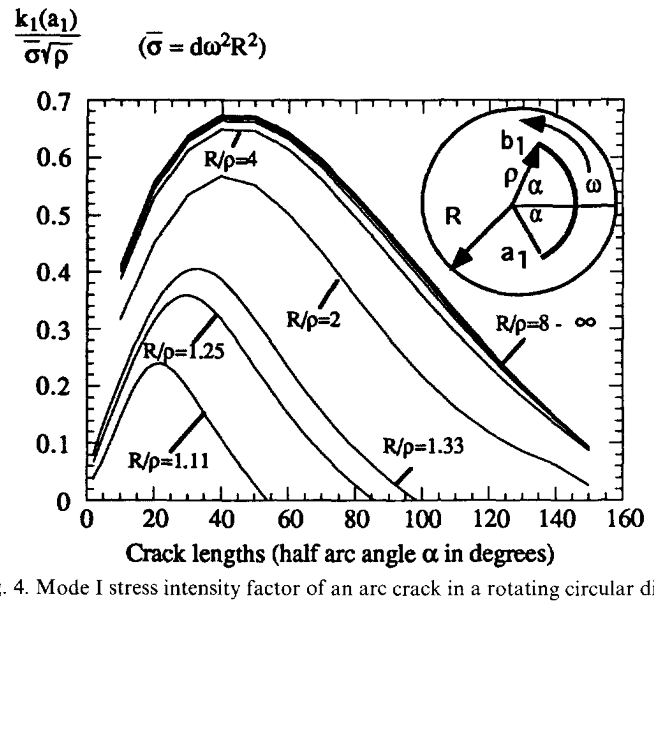

Fig. 4. Mode I stress intensity factor of an arc crack in a rotating circular disk.

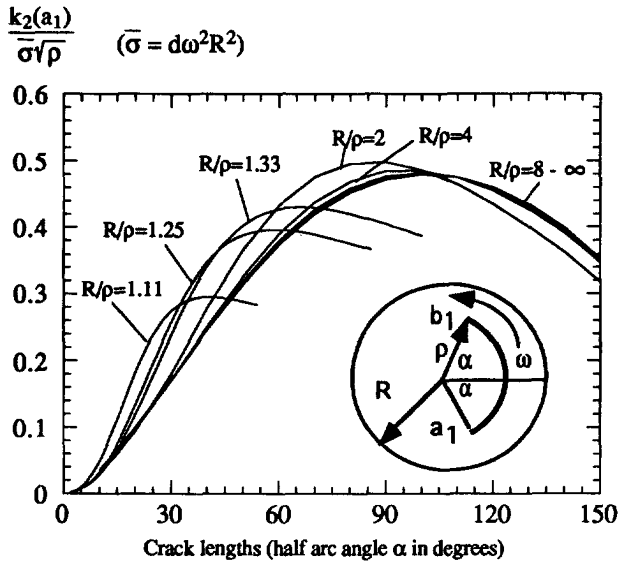

Fig. 5. Mode II stress intensity factor of an arc crack in a rotating circular disk.

$$K_3(s,t) = -\left[\frac{s}{4}\left(\frac{1-s^2}{1+s^2}+\frac{1-t^2}{1+t^2}\right)+\frac{1}{2}\left(\frac{s}{1+s^2}-\frac{t}{1+t^2}\right)\right] \tag{61}$$

$$K_4(s,t) = \left[s\left(\frac{s}{1+s^2}+\frac{t}{1+t^2}\right)-\frac{1}{2}\left(\frac{1-s^2}{1+s^2}-\frac{1-t^2}{1+t^2}\right)\right]. \tag{62}$$

For an infinite plane containing an arc crack subjected to uniform biaxial tractions at infinity, Sih (1973) has obtained explicit formulas for calculating the stress intensity factors. It may be seen in Table 1, that the present results match the exact solution exceptionally well.

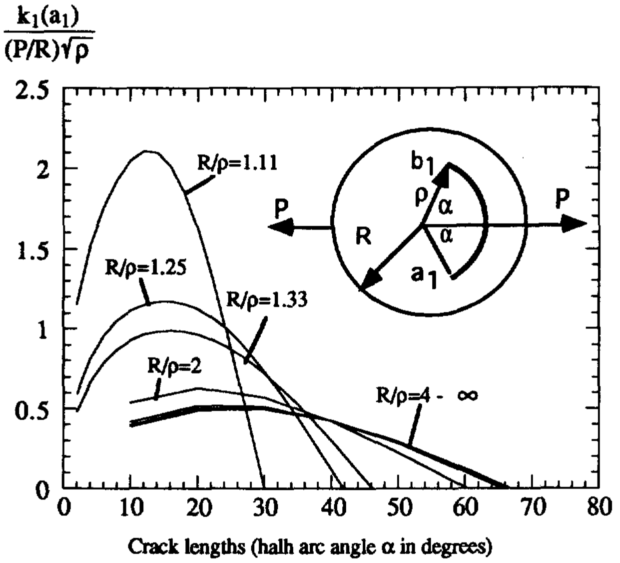

Fig. 6. Mode I stress intensity factor of an arc crack in a circular disk subjected to point forces.

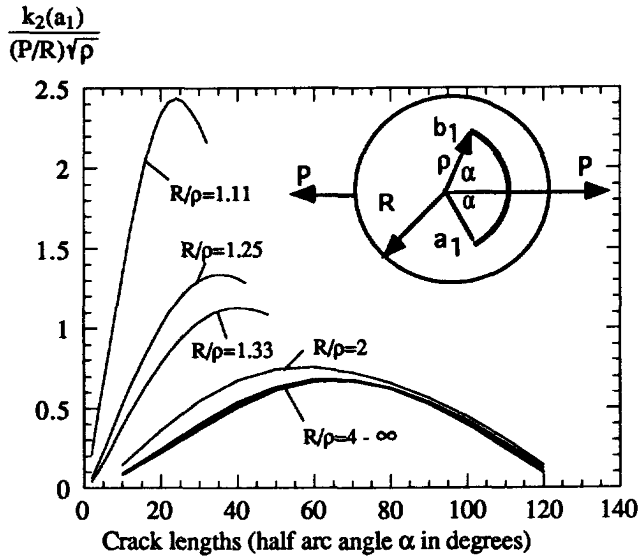

Fig. 7. Mode II stress intensity factor of an arc crack in a circular disk subjected to point forces.

Table 1. Normalized stress intensity factors of an arc crack in an infinite plane subjected to uniform biaxial traction at infinity

	$\frac{k_1(\rho\alpha)}{q_0\sqrt{\rho}}$		$\frac{k_2(\rho\alpha)}{q_0\sqrt{\rho}}$	
α (degrees)	Present solution	Sih's solution	Present solution	Sih's solution
10	0.4120	0.4120	0.0360	0.0360
20	0.5591	0.5591	0.0986	0.0986
30	0.6401	0.6401	0.1715	0.1715
40	0.6745	0.6745	0.2455	0.2455
50	0.6730	0.6730	0.3138	0.3138
60	0.6447	0.6447	0.3722	0.3722
70	0.5975	0.5975	0.4184	0.4184
80	0.5379	0.5379	0.4514	0.4514
90	0.4714	0.4714	0.4714	0.4714
100	0.4020	0.4020	0.4791	0.4791
110	0.3327	0.3327	0.4752	0.4752
120	0.2659	0.2659	0.4605	0.4605
130	0.2031	0.2031	0.4355	0.4355
140	0.1456	0.1456	0.4001	0.4001
150	0.0935	0.0947	0.3535	0.3533

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APPENDIX

Dislocation solution

The geometry of the dislocation case is a circular disk ($r \leqslant R$) containing an edge dislocation with Burgers vector $(b_x, b_y, 0)$ at $z_0 = \rho e^{i\alpha}$. The disk is free from tractions at $r = R$ and the boundary condition may be written as,

$$\sigma_{rr} + i\tau_{r\theta} = 0, \quad r = R, \quad 0 \leqslant \theta \leqslant 2\pi. \tag{A.1}$$

Using Muskhelishvili's complex variable technique (1953b), the stresses and displacements in a two-dimensional elasticity case may be represented in terms of two complex potentials, $\Phi(z)$ and $\Psi(z)$, as follows,

$$\sigma_{rr} + \sigma_{\theta\theta} = 4\,\mathrm{Re}\{\Phi(z)\} \tag{A.2}$$

$$\sigma_{rr} - \sigma_{\theta\theta} + 2i\tau_{r\theta} = 2\frac{z}{\bar{z}}[\bar{z}\Phi'(z) + \Psi(z)] \tag{A.3}$$

$$2\mu(u_r + u_\theta) = e^{i\theta}[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}] \tag{A.4}$$

with

$$\Phi(z) = \varphi'(z), \quad \Psi(z) = \psi'(z), \tag{A.5}$$

where the prime denotes the derivative with respect to z, an overbar represents the complex conjugate and Re{ } denotes the real part of the expression in parentheses; $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for the plane stress, with ν being the Possion's ratio and μ the shear modulus.

Combining eqns (A.2) and (A.3), then taking the complex conjugate yields

$$\sigma_{rr} + i\tau_{r\theta} = \Phi(z) + \overline{\Phi(z)} - \frac{\bar{z}}{z}[z\overline{\Phi'(z)} + \overline{\Psi(z)}]. \tag{A.6}$$

Due to the existence of the edge dislocation in the disk, the complex potentials are constructed in the following forms:

$$\Phi(z) = \Phi_0(z) + \Phi_1(z) \tag{A.7}$$

$$\Psi(z) = \Psi_0(z) + \Psi_1(z), \tag{A.8}$$

where $\Phi_0(z)$ and $\Psi_0(z)$ are the stress functions for an edge dislocation with Burgers vector $(b_x, b_y, 0)$ embedded at $z_0 = \rho e^{i\alpha}$ in an infinite plane, which are known as (Muskhelishvili, 1953b),

$$\Phi_0(z) = \frac{A}{z - z_0} \tag{A.9}$$

$$\Psi_0(z) = \frac{\bar{A}}{z - z_0} + \frac{A\bar{z}_0}{(z - z_0)^2}, \tag{A.10}$$

where

$$A = \frac{\mu(b_y - ib_x)}{\pi(1+\kappa)}; \tag{A.11}$$

$\Phi_1(z)$ and $\Psi_1(z)$ are non-singular parts of the potentials, and are to be determined in such a way that the boundary conditions can be satisfied.

Combining eqns (A.6), (A.7), (A.8), (A.9) and (A.10), we may obtain

$$\Phi_1(t) + \overline{\Phi_1(t)} - \frac{R^2}{t}\overline{\Phi_1'(t)} - \frac{R^2}{t^2}\overline{\Psi_1(t)} = g(t), \quad t = Re^{i\theta}, \quad 0 \leqslant \theta \leqslant 2\pi \tag{A.12}$$

where

$$g(t) = -\frac{A}{t - z_0} - \frac{\bar{A}t}{R^2 - \bar{z}_0 t} - \frac{\bar{A}R^2 t}{(R^2 - \bar{z}_0 t)^2} + \frac{AR^2}{t(R^2 - \bar{z}_0 t)} + \frac{\bar{A}R^2 z_0}{(R^2 - \bar{z}_0 t)^2}. \tag{A.13}$$

We may extend $\Phi_1(z)$ to the outside of the disk by defining

$$\Phi_1\left(\frac{R^2}{\bar{z}}\right) = -\overline{\Phi_1(z)} + \bar{z}\overline{\Phi_1'(z)} + \frac{\bar{z}^2}{R^2}\overline{\Psi_1(z)}, \quad (z \in S_{\text{int}}) \tag{A.14}$$

or equivalently,

$$\Phi_1(z) = -\bar{\Phi}_1\left(\frac{R^2}{z}\right) + \frac{R^2}{z}\bar{\Phi}_1'\left(\frac{R^2}{z}\right) + \frac{R^2}{z^2}\bar{\Psi}_1\left(\frac{R^2}{z}\right)^{\dagger} \quad (z \in S_{\text{ext}}). \tag{A.15}$$

For eqn (A.14), we may also write

$$\Psi_1(z) = \frac{R^2}{z^2}\Phi_1(z) - \frac{R^2}{z}\Phi_1'(z) + \frac{R^2}{z^2}\bar{\Phi}_1\left(\frac{R^2}{z}\right), \quad (z \in S_{\text{int}}). \tag{A.16}$$

Since $\Phi_1(z)$ and $\Psi_1(z)$ are holomorphic in the disk domain, from the definition in eqn (A.15), $\Phi_1(z)$ should also be holomorphic outside the disk, including at infinity, where its principal part is a complex constant.

Combining eqns (A.6), and (A.16), and imposing the boundary condition (A.12), it follows that

$$\Phi_1^+(t) - \Phi_1^-(t) = g(t), \quad t = Re^{i\theta}, \quad 0 \leqslant \theta \leqslant 2\pi, \tag{A.17}$$

where "+" and "−" denote approach to the boundary, $r = R$, from outside and inside the circle, respectively.

The solution to eqn (A.17) can be found using the Plemelj formula (Muskhelishvili, 1953a), i.e.

$$\Phi_1(z) = \frac{1}{2\pi i}\int_c \frac{g(t)}{t - z}\,dt + c_1. \tag{A.18}$$

It turns out that

$$\Phi_1(z) = \begin{cases} \dfrac{\bar{A}}{\bar{z}_0} + \dfrac{A\bar{z}_0}{R^2 - \bar{z}_0 z} - \dfrac{\bar{A}R^2}{(R^2 - \bar{z}_0 z)^2}\left(\dfrac{R^2}{\bar{z}_0} - z_0\right) + c_1, & (z \in S_{\text{int}}) \\ \dfrac{A}{z - z_0} - \dfrac{A}{z} + c_1, & (z \in S_{\text{ext}}), \end{cases} \tag{A.19}$$

where c_1 is a complex constant. Substituting eqn (A.19) into (A.16) and using the fact that $\Psi_1(z)$ is holomorphic in the disk domain, we may obtain

$$\text{Re}\{c_1\} = -\frac{\mu(b_y\rho\cos\alpha - b_x\rho\sin\alpha)}{\pi(1+\kappa)R^2}. \tag{A.20}$$

Since the complex potential $\Phi_1(z)$ can differ in the imaginary number without causing any change in the stresses, we may simply let the imaginary part of c_1 be zero.

Substituting eqn (A.19) into eqn (A.16) yields $\Psi_1(z)$ as below:

$$\Psi_1(z) = \frac{\bar{A}\bar{z}_0}{R^2 - \bar{z}_0 z} + \frac{\bar{A}(R^2 - \bar{z}_0 z_0)\bar{z}_0 - A\bar{z}_0^3}{(R^2 - \bar{z}_0 z)^2} + \frac{2\bar{A}R^2\bar{z}_0(R^2 - z_0\bar{z}_0)}{(R^2 - \bar{z}_0 z)^3}, \quad (z \in S_{\text{int}}). \tag{A.21}$$

With the complex potentials given in eqns (A.9), (A.10), (A.19) and (A.21), the stresses can be readily obtained from eqns (A.2) and (A.3). Some of the results are given in eqns (A.22)–(A.27) as below, and the displacements may then be obtained using the constitutive and kinematic relationships.

† $\overline{f(z)} = \bar{f}(\bar{z})$; to obtain $\bar{f}(R^2/z)$, simply write R^2/z for $\bar{z}$ in function $\bar{f}(\bar{z})$.

(a) $b_x \neq 0,\ b_y = 0,$

$$\sigma_{rr}(r,\theta) = \frac{\mu b_x}{\pi(1+\kappa)}\left\{\frac{2\rho\sin\alpha - r\sin\theta + \rho\sin(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{-2\rho R^2\sin\alpha + r\rho^2\sin\theta - \rho R^2\sin(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)}\right.$$
$$-\frac{3r\rho^2\sin(2\alpha-\theta) - 3\rho r^2\sin\alpha - \rho^3\sin(3\alpha-2\theta) + r^3\sin\theta}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2} + \frac{2\rho\sin\alpha}{R^2}$$
$$-\frac{R^4\rho(R^2-\rho^2-2r^2)\sin(\alpha-2\theta) - R^2r\rho^2(4r^2+5R^2+2\rho^2)\sin(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$-\frac{R^2\rho^3(2r^2+R^2)\sin(3\alpha-2\theta) + \rho(2R^6+2r^4\rho^2+8r^2R^4+r^2\rho^2R^2+2r^2\rho^4)\sin\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$+\frac{r(4R^6+4\rho^2r^2R^2+\rho^4r^2-2\rho^2R^4+2\rho^4R^2)\sin\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$-\frac{2R^2(R^2-\rho^2)[\rho R^6\sin(\alpha-2\theta) - r^4\rho^3\sin(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$
$$\left.-\frac{2R^2(R^2-\rho^2)[R^4r(3\rho^2+R^2)\sin\theta - 3R^2\rho r^2(R^2+\rho^2)\sin\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}\right\} \tag{A.22}$$

$$\tau_{r\theta}(r,\theta) = \frac{\mu b_x}{\pi(1+\kappa)}\left\{\frac{r\cos\theta - \rho\cos(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} + \frac{\rho R^2\cos(\alpha-2\theta) - r\rho^2\cos\theta}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)}\right.$$
$$+\frac{r^3\cos\theta + 3r\rho^2\cos(2\alpha-\theta) - 3\rho r^2\cos\alpha - \rho^3\cos(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2}$$
$$-\frac{R^4\rho(\rho^2-R^2)\cos(\alpha-2\theta) + R^2r\rho^2(R^2+2\rho^2)\cos(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$-\frac{-\rho^3R^4\cos(3\alpha-2\theta) + r\rho^2(r^2\rho^2+2R^4-2R^2\rho^2)\cos\theta - 3r^2R^2\rho^3\cos\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$-\frac{2R^2(R^2-\rho^2)[-\rho R^6\cos(\alpha-2\theta) - r^4\rho^3\cos(3\alpha-2\theta) + r^3\rho^2(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$
$$\left.-\frac{2R^2(R^2-\rho^2)[R^4r(3\rho^2+R^2)\cos\theta - 3R^2\rho r^2(R^2+\rho^2)\cos\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}\right\}. \tag{A.23}$$

(b) $b_x = 0,\ b_y \neq 0,$

$$\sigma_{rr}(r,\theta) = \frac{\mu b_y}{\pi(1+\kappa)}\left\{\frac{r\cos\theta - 2\rho\cos\alpha + \rho\cos(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)}\right.$$
$$-\frac{3\rho r^2\cos\alpha - r^3\cos\theta - 3\rho^2r\cos(2\alpha-\theta) + \rho^3\cos(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2}$$
$$+\frac{2\rho R^2\cos\alpha - \rho^2r\cos\theta - \rho R^2\cos(\alpha-2\theta)}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)} - \frac{2\rho\cos\alpha}{R^2}$$
$$+\frac{\rho R^4(\rho^2-R^2+2r^2)\cos(\alpha-2\theta) - R^2r\rho^2(4r^2+5R^2+2\rho^2)\cos(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$+\frac{R^2\rho^3(2r^2+R^2)\cos(3\alpha-2\theta) + \rho(2R^6+2r^4\rho^2+8r^2R^4+r^2\rho^2R^2+2r^2\rho^4)\cos\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$
$$-\frac{r(4R^6+4\rho^2r^2R^2+\rho^4r^2-2\rho^2R^4+2\rho^4R^2)\cos\theta}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$
$$-\frac{2R^2(R^2-\rho^2)[\rho R^6\cos(\alpha-2\theta) + r^4\rho^3\cos(3\alpha-2\theta) - r^3\rho^2(3R^2+\rho^2)\cos(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$
$$\left.-\frac{2R^4r(R^2-\rho^2)[3\rho r(R^2+\rho^2)\cos\alpha - R^2(3\rho^2+R^2)\cos\theta]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}\right\} \tag{A.24}$$

$$\tau_{r\theta}(r,\theta) = \frac{\mu b_y}{\pi(1+\kappa)}\left\{\frac{r\sin\theta+\rho\sin(\alpha-2\theta)}{r^2+\rho^2-2r\rho\cos(\theta-\alpha)} - \frac{\rho R^2\sin(\alpha-2\theta)+r\rho^2\sin\theta}{R^4+r^2\rho^2-2R^2r\rho\cos(\theta-\alpha)}\right.$$

$$+\frac{r^3\sin\theta+3r\rho^2\sin(2\alpha-\theta)-3\rho r^2\sin\alpha-\rho^3\sin(3\alpha-2\theta)}{[r^2+\rho^2-2r\rho\cos(\theta-\alpha)]^2}$$

$$-\frac{\mathrm{R}^4\rho(\mathrm{R}^2-\rho^2)\sin(\alpha-2\theta)+R^2r\rho^2(R^2+2\rho^2)\sin(2\alpha-\theta)}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$-\frac{-\rho^3R^4\sin(3\alpha-2\theta)+r\rho^2(r^2\rho^2+2R^4-2R^2\rho^2)\sin\theta-3r^2R^2\rho^3\sin\alpha}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^2}$$

$$-\frac{2R^2(R^2-\rho^2)[pR^6\sin(\alpha-2\theta)-r^4\rho^3\sin(3\alpha-2\theta)+r^3\rho^2(3R^2+\rho^2)\sin(2\alpha-\theta)]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}$$

$$\left.-\frac{2R^4r(R^2-\rho^2)[R^2(3\rho^2+R^2)\sin\theta-3\rho r(R^2+\rho^2)\sin\alpha]}{[R^4+r^2\rho^2-2r\rho R^2\cos(\theta-\alpha)]^3}\right\}. \tag{A.25}$$